Balancing And Lucas-balancing Numbers With Real Indices

A thesis

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Under the supervision of

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DECLARATION

I here by declare that the topic "Balancing and Lucas balancing numbers for real indices" as a partial fulfillment of my M.Sc. degree has not been submitted to any other institution or the university for the award of any other degree or diploma.

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CERTIFICATE

This is to certify that the project report entitled **Balancing and Lucas-balancing numbers with real indices** submitted by **Sephali Tanty** to the National Institute of Technology, Rourkela, Odisha for the partial fulfilment of requirements for the degree of Master of Science in Mathematics is a bonafide record of work carried out by her under my supervision and guidance. The contents of the project, in full or in parts have not been submitted to any other institution or the university or the award of any other Degree or Diploma.

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ABSTRACT

In this thesis, we have studied the balancing and Lucas-balancing numbers for real indices. Also we have discussed some properties of balancing numbers with real numbers. We extend some properties of Fibonacci numbers to balancing numbers.

Contents

CHAPTER-1

1. INTRODUCTION

There is a famous quote by the famous Johann Carl Friedrich Gauss a German Mathematician $(30\text{ April}, 1777 - 23\text{ February}, 1855)$: "Mathematics is the queen of all Sciences, and Number theory is the queen of Mathematics", (see [4, 5]). Number theory, or higher arithmetics is the study of those properties of integers which we used everyday arithmetic.

In Number theory (see [4, 5]), discovery of number sequences with certain specified properties has been a source of attraction since ancient times. The most beautiful and simplest of all number sequences is the Fibonacci sequence. This sequence was first invented by Leonardo of Pisa $(1170 - 1250)$, who was also known as Fibonacci, to describe the growth of a rabbit population. The Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \cdots$ and denoted by $F_n,\ n \geq 1$.

Generally known, Number Theory as the study of properties of numbers. But now-a-days the application of Number Theory uses in diverse fields like, Coding Theory, Cryptology (the art of creating and breaking codes), Computer Science.

Other interesting number sequences are Pell's sequence and the Associated Pell's sequence, (see [4, 5]). In Mathematics, the Pell's numbers are infinite sequence of integers that have been known since ancient times. The denominators of the closest rational approximations to the square root of 2. This sequence begins with $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$; so the sequences of Pell's numbers begin with $1, 2, 5, 12, 29$. The numerators of the same sequence of approximations give the Associated Pell sequence.

The concept of balancing numbers was first invented by Behera and Panda, (see [1]) in the year 1999 in connection with a Diophantine equation. It consists of finding a natural number n such that:

$$1 + 2 + 3 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some natural number r, is balancer corresponding to the balancing number n. If the n^{th} triangular number $\frac{n(n+1)}{2}$ is denoted by T_n, then the above equations reduces to $T_{n-1}+T_n = T_{n+r}$, which is the problem finding two consecutive triangular numbers whose sum is also a triangular number. Since,

$$T_5 + T_6 = 15 + 21 = 36 = T_8$$

6 is a balancing number with balancer 2. Similarly,

$$T_{34} + T_{35} = 595 + 630 = 1225 = T_{49}$$

implies that, 35 is also a balancing number with balancer 14. The balancing numbers, though obtained from a simple Diophantine equation, are very useful for the computation of square triangular numbers. An important result about balancing numbers is that, n is a balancing number if and only if $8n^2 + 1$ is a perfect square, and the number $\sqrt{8n^2+1}$ is called a Lucas- balancing numbers is that, these numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers, (see [1, 2]).

There are some other properties common to both Fibonacci as well as balancing numbers. As shown by Behera and Panda, (see [1, 10]), the square of balancing numbers is is a triangular number.The search for balancing numbers in well known integer sequences was first initiated by Liptai, (see [2]). He proved that, there is no balancing number in the Fibonacci sequences other than 1.

CHAPTER-2

2. Preliminaries

In this chapter, we recall some definitions known results of Fibonacci and Lucas numbers, Balancing and Lucas-balancing numbers, Pell's numbers, Associated Pell's numbers, Real numbers, Recurrence relations, Binet formula, Diophantine equations, complex function.

2.1. Recurrence relations

In Mathematics, a recurrence relations is an equation that defines a sequences recursively, each term of the sequences is defined as a functions of the preceding terms, (see [4, 5]).

2.2. Fibonacci sequences

The Fibonacci numbers are defined recursively as $F_1 = 1,\ F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 2.$$

(see [4, 5]).

2.3. Lucas sequences

Lucas sequences is also obtained from the same recurrence relation as that for Fibonacci numbers. The Lucas numbers are defined recursively as $L_1 = 1,\ L_2 = 3$ and

$$L_{n+1} = L_n + L_{n-1}, \quad n \geq 2.$$

(see [4, 5]).

2.4. Diophantine equations

In Mathematics, a Diophantine equations is an intermediate polynomial equations that allows the variables to be integers only. Diophantine problems have fewer equations than unknowns and involve finding integers that work correctly for all the equations, (see [4, 5, 9]).

2.5. Binet Formula

While solving a recurrence relations, as a difference equations, the n^{th} term of the sequence is obtained in closed form, which is a equation containing conjugate surds of irrational number is known as the *Binet Formula* for the particular sequence. The Binet Formula for the Fibonacci sequences is,

$$F_n = \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1}.$$

The Binet Formula for the Lucas sequences is given by,

$$L_n = \alpha_1^n + \beta_1^n.$$

where $n \in N$, $\alpha_1 = \frac{1+\sqrt{5}}{2}$, $\beta_1 = \frac{1-\sqrt{5}}{2}$,
(see [1, 4, 5, 9]).

2.6. Pell's numbers

The Pell's numbers are defined recursively as $P_1 = 1,\ P_2 = 2$ and

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2.$$

(see [4, 5]),

1. Cassini Formula: $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$
2. Binet Formula: $\frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2}$ where, $\alpha_2 = 1 + \sqrt{2}$ and $\beta_2 = 1 - \sqrt{2}$,

and

$$P_n = 1, 2, 5, 12, 29, \cdots, \quad n = 1, 2, \cdots$$

2.7. Associated Pell's numbers

The Associated Pell's numbers are defined recursively as $Q_1 = 1,\ Q_2 = 2$ and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2.$$

(see [4, 5, 10]),

1. Cassini Formula: $Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n$.
2. Binet Formula: $\frac{\alpha_2^n - \beta_2^n}{\alpha_2 + \beta_2}$ where, $\alpha_2 = 1 + \sqrt{2}$ and $\beta_2 = 1 - \sqrt{2}$,

and

$$Q_n = 1, 3, 7, 17, \cdots, \quad n = 1, 2, \cdots$$

2.8. Balancing Numbers

The balancing numbers are defined recursively as $B_1 = 1,\ B_2 = 6$ and

$$B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 2.$$

(see [1, 2, 9, 10]),

1. Cassini Formula: $B_n^2 - B_{n+1}B_{n-1} = 1$.
2. Binet Formula: $\frac{\alpha^n - \beta^n}{\alpha - \beta}$ where, $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$, and

$$B_n = 1, 6, 35, 204, 1189, \cdots, \quad n = 1, 2, \cdots$$

2.9. Lucas-Balancing Numbers

The Lucas-balancing numbers are defined recursively as $C_1 = 3,\ C_2 = 17$ and

$$C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2.$$

(see [1, 2, 9, 10]),

1. Cassini Formula: $C_n^2 - C_{n+1}C_{n-1} = -8$.
2. Binet Formula: $\frac{\alpha^n - \beta^n}{2}$ where, $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$, and

$$C_n = 3, 17, 99, 577, 3363 \cdots, \quad n = 1, 2, \cdots$$

CHAPTER-3

3. Balancing and Lucas-balancing with real indices

R. Witula, (see [12]) discovered some new formulas for Fibonacci and Lucas numbers by using real numbers.

3.1. Introduction

Usually, every number sequence is defined for integral indices. In a recent paper, (see [12]), Witula introduced Fibonacci and Lucas numbers with real indices and provided some applications. We devote this chapter to explain the work of Witula,(see [12]).

Using the Binet Formulas for Fibonacci and Lucas numbers, (see [1, 4, 5]), one can easily get,

1. $\sqrt{5}F_n = \alpha_1^n - \beta_1^n$,
2. $F_{n+1} - \beta_1 F_n = \alpha_1^n$,
3. $L_n = 2\alpha_1^n - \sqrt{5}F_n$.

Using the Binet formulas of balancing and Lucas-balancing numbers the following results can be obtained.

$$2\sqrt{8}B_n = \alpha^n - \beta^n, \tag{1}$$

$$C_n = \alpha^n - \sqrt{8}B_n. \tag{2}$$

It is easy to see that,

$$B_{n+1} - \beta B_n = \alpha^n. \tag{3}$$

where, $\alpha = 3 + \sqrt{8}$ and $\beta = 3 - \sqrt{8}$.

3.2. Fibonacci and Lucas numbers with real indices

These following results were developed by the R.Witula, (see [12]).
Let s be real number. Then,

1. $\sqrt{5}F_s = \alpha_1^s - e^{i\pi s}(-\beta_1^s)$,

2. $F_{s+1} = \alpha_1^s + \beta_1 F_s$,

3. $L_s = 2\alpha_1^s + \sqrt{5}F_s$.

where $\alpha_1,\ \beta_1 > 0$ and $e^{i\pi s} = \cos\pi s + i\sin\pi s$, is a complex function. Then,

$$\begin{aligned} F_{s+1} &= \alpha_1^s + \beta_1 F_s \\ &= \sqrt{5}F_s + e^{i\pi s}(-\beta_1)^s + \beta_1 F_s. \end{aligned}$$

Hence,

$$F_{s+1} = e^{i\pi s}(-\beta_1)^s + \alpha_1 F_s.$$

3.3. Balancing and Lucas-balancing numbers with real indices

Let s be real number. We define balancing numbers B_s and Lucas-balancing numbers C_s with real indices as,

$$2\sqrt{8}B_s = \alpha^s - e^{i\pi s}(-\beta)^s, \tag{4}$$

$$C_s = \alpha^s - \sqrt{8}B_s. \tag{5}$$

It is easy to see that,

$$B_{s+1} = \alpha^s + \beta B_s. \tag{6}$$

where $\alpha,\ \beta > 0$ and $e^{i\pi s}$ is a complex function.

Note that, (4) and (6) imply,

$$\begin{aligned} B_{s+1} &= \alpha^s + \beta B_s \\ &= 2\sqrt{8}B_s + e^{i\pi s}(-\beta)^s + \beta B_s. \end{aligned}$$

Hence,

$$B_{s+1} = e^{i\pi s}(-\beta)^s + \alpha B_s. \tag{7}$$

3.4. Some Properties of Fibonacci and Lucas numbers

The following are some properties related to F_s and L_s , $s \in R$, (see [12]).

1. $\sqrt{5}F_s = \alpha^s - e^{i\pi s}(-\beta)^s$.
2. $L_s = \alpha^s + e^{i\pi s}(-\beta)^s$.
3. $F_{s+2} = F_{s+1} + F_s$.
4. $L_{s+2} = L_{s+1} + L_s$.
5. $F_{-s} = -e^{-i\pi s}F_s$.
6. $L_{-s} = e^{-i\pi s}L_s$.
7. $F_{2s} = F_s L_s$.
8. $L_s = F_{s+1} + F_{s-1}$.
9. $5F_s^2 = L_{2s} - 2e^{i\pi s}$.
10. $5F_s^3 = F_{3s} - 33e^{i\pi s}F_s$.
11. $F_{s+t+1} = F_{s+1}F_{t+1} - F_s F_t$.
12. $e^{i\pi t}L_{s-t} = F_{t+1}L_s - F_t L_{s+1}$.
13. $25F_s^4 = L_{4s} - 4e^{i\pi s}L_{2s} + 6e^{3i\pi s}$.
14. $F_s F_t - F_{s-r}F_{t+r} = e^{i\pi(s-r)}F_r F_{t-s+r}$.

3.5. Some Properties of balancing and Lucas-balancing numbers

We prove the following properties of B_s and C_s with s $\in R$.

$$2C_s = \alpha^s + e^{i\pi s}(-\beta)^s. \tag{8}$$

$$B_{s+2} = 6B_{s+1} - B_s. \tag{9}$$

$$C_{s+2} = 6C_{s+1} - C_s. \tag{10}$$

$$B_{-s} = \pm e^{-i\pi s}B_s = \begin{cases} e^{i\pi s}B_s, & \text{if } s \text{ is odd;} \\ -e^{i\pi s}B_s, & \text{if } s \text{ is even.} \end{cases} \tag{11}$$

$$C_{-s} = \pm e^{-i\pi s}C_s = \begin{cases} -e^{i\pi s}C_s, & \text{if } s \text{ is odd;} \\ e^{i\pi s}C_s, & \text{if } s \text{ is even.} \end{cases} \tag{12}$$

$$B_{2s} = 2B_sC_s. \tag{13}$$

$$2C_s = B_{s+1} - B_{s-1}. \tag{14}$$

$$\pm\, e^{i\pi t}C_{s-t} = B_{t+1}C_s - B_tC_{s+1} = \begin{cases} e^{i\pi t}, & \text{if } s \text{ is even;} \\ -e^{i\pi t}, & \text{if } s \text{ is odd.} \end{cases} \tag{15}$$

$$B_sB_t - B_{s-r}B_{t+r} = e^{i\pi(s-r)}B_rB_{t-s+r}. \tag{16}$$

$$B_{s+t+1} = B_{s+1}B_{t+1} - B_sB_t. \tag{17}$$

$$16B_s^2 = C_{2s} \pm e^{i\pi s} = \begin{cases} e^{i\pi s}, & \text{if } s \text{ is odd;} \\ -e^{i\pi s}, & \text{if } s \text{ is even.} \end{cases} \tag{18}$$

$$32B_s^3 = B_{3s} \pm 3e^{i\pi s}B_s = \begin{cases} 3e^{i\pi s}B_s, & \text{if } s \text{ is odd;} \\ -3e^{i\pi s}B_s, & \text{if } s \text{ is even.} \end{cases} \tag{19}$$

3.6. Proofs of the Properties for balancing and Lucas-balancing numbers

Using equations (6), we get (9) as follows.

$$B_{s+2} = 6B_{s+1} - B_s.$$

$$\begin{aligned} B_{s+2} &= \alpha^{s+1} + \beta(\alpha^s + \beta B_s) \\ &= 6\alpha^s + \beta^2 B_s \\ &= 6\alpha^s + 6\beta B_s - B_s \\ &= 6B_{s+1} - B_s, \end{aligned}$$

The proof of (10) is similar to that of (9).

Using (4) and the fact that $\alpha\beta = 1$, (11) we have,

$$B_{-s} = \pm e^{-i\pi s} B_s.$$

$$\begin{aligned} 2\sqrt{8}B_{-s} &= \alpha^{-s} - e^{-i\pi s}(-\beta)^s \\ &= \beta^s - e^{-i\pi s}(-\alpha)^s \\ &= [e^{i\pi s}(\beta)^s - (-1)^s\alpha^s]/e^{i\pi s} \\ &= [(-1)^{2s}e^{i\pi s}(\beta)^s - (-1)^s\alpha^s]/e^{i\pi s} \\ &= (-1)^{s+1}[\alpha^s - (-1)^s e^{i\pi s}(\beta)^s]e^{-i\pi s} \\ &= (-1)^{s+1}[\alpha^s - e^{i\pi s}(\beta)^s]e^{-i\pi s} \\ &= (-1)^{s+1}2\sqrt{8}B_s e^{-i\pi s}, \end{aligned}$$

The proof of (12) is same as that of (11).

Putting the value of (4) and (8), we get (13) as follows.

$$B_{2s} = 2B_sC_s.$$

$$\begin{aligned} 2\sqrt{8}B_{2s} &= \alpha^{2s} - e^{2i\pi s}(-\beta)^s] \\ &= 2C_s[\alpha^s - (-1)^s e^{i\pi s}(\beta)^s] \\ &= 2C_s 2\sqrt{8}B_s. \end{aligned}$$

Using (5), we get (14) as follows.

$$2C_s = B_{s+1} - B_{s-1}.$$

$$\begin{aligned} C_s &= \alpha^s - \sqrt{8}B_s \\ &= [2\alpha^s + \beta B_s - \alpha B_s]/2 \\ &= [B_{s+1} + \alpha(\alpha^{s-1} - B_s)]/2 \\ &[\because B_s = \alpha^{s-1} + B_{s-1}] \\ &= [B_{s+1} - B_{s-1}]/2. \end{aligned}$$

Putting the value of (4) and (7), we get (15) as follows.

From

$$\begin{aligned} & e^{i\pi t}\alpha^s(-\beta)^t + e^{i\pi s}\alpha^t(-\beta)^s \\ &= e^{i\pi t}\alpha^t(-\beta^t)[\alpha^{s-t} + e^{i\pi(s-t)}(-\beta^{s-t})] \\ &= (-1)^t e^{i\pi t}(2C_{s-t}). \end{aligned}$$

and

$$\begin{aligned}
&e^{i\pi t}\alpha^s(-\beta^t) + e^{i\pi s}\alpha^t(-\beta^s) \\
&= e^{i\pi t}(-\beta^t)(B_{s+1} - \beta B_s) + \alpha^t(B_{s+1} - \alpha B_s) \\
&= (B_{t+1} - \alpha B_t)(B_{s+1} - \beta B_s) + (B_{t+1} - \beta B_t)(B_{s+1} - \alpha B_s) \\
&= B_{t+1}(2B_{s+1} - 6B_s) + B_t(2B_s - 6B_{s+1}) \\
&= B_{t+1}(2B_{s+1} - B_{s+1} - B_{s-1}) + B_t(2B_s - B_{s+2} - B_s) \\
&= B_{t+1}2C_s - B_t 2C_{s+1}.
\end{aligned}$$

(15) follows.

Using (4), we get (16) as follows.

$$B_sB_t - B_{s-r}B_{t+r} = e^{i\pi(s-r)}B_rB_{t-s+r}.$$

$$\begin{aligned}
&B_sB_t - B_{s-r}B_{t+r} \\
&= [-e^{i\pi t}(-\beta^t)\alpha^s - e^{i\pi s}(-\beta^s)\alpha^t + e^{i\pi(t+r)}(-\beta^{t+r})\alpha^{s-r} \\
&+ e^{i\pi(s-r)}(-\beta^{s-r})\alpha^{t+r}]/32 \\
&= B_r[e^{i\pi(s-r)}(-\beta^{s-r})\alpha^{s-r}(\alpha^{t-s+r} - e^{i\pi(t-s+r)}(-\beta^{t-s+r})] \\
&= e^{i\pi(s-r)}B_rB_{t-s+r}.
\end{aligned}$$

Again from (4), we get (17) as follows.

$$B_{s+t+1} = B_{s+1}B_{t+1} - B_sB_t.$$

$$\begin{aligned}
&B_{s+1}B_{t+1} - B_sB_t \\
&\qquad = [\alpha^{s+t+1}(\alpha - \alpha^{-1}) + e^{i\pi(s+t+1)}(-\beta)^{s+t+1}(\beta - e^{-i\pi}(-\beta)^{-1})]/32 \\
&\qquad = [2\sqrt{8}\alpha^{s+t+1} - 2\sqrt{8}e^{i\pi(s+t+1)}(-\beta)^{s+t+1}]/32 \\
&\qquad = B_{s+t+1}.
\end{aligned}$$

Using (4) and (5), we get (18) as follows.

$$16B_s^2 = C_{2s} \pm e^{i\pi s}.$$

$$\begin{aligned}[2\sqrt{8}B_s 2\sqrt{8}B_s]/2 &= [\alpha^{2s} + e^{2i\pi s}(-\beta^{2s}) - 2e^{i\pi s}\alpha^s(-\beta^s)]/2 \\ &= C_{2s} - e^{i\pi s}(-1)^s \\ &= C_{2s} + (-1)^{s+1}e^{i\pi s}.\end{aligned}$$

Using (4), we get (19) as follows.

$$32B_s^3 = B_{3s} \pm 3e^{i\pi s}B_s.$$

$$\begin{aligned}32B_s^3 &= [2\sqrt{8}B_s]^3/2\sqrt{8} \\ &= [\alpha^{3s} - e^{3i\pi s}(-\beta^{3s}) - 3e^{i\pi s}\alpha^s(-\beta^s)[\alpha^s - e^{i\pi s}(-\beta^s)]]/2\sqrt{8} \\ &= [2\sqrt{8}B_{3s} - 3e^{i\pi s}(-1)^s 2\sqrt{8}B_s]2\sqrt{8} \\ &= B_{3s} + (-1)^{s+1}3e^{i\pi s}B_s.\end{aligned}$$

References

[1] A. Behera and G.K. Panda, "*On the Square roots of Triangular numbers*", Fibonacci Quarterly, 37 No. 2, 1999, 98-105.

[2] Berczes, A., K. Liptai, I. Pink, Fibonacci Quarterly, "*On Generalized Balancing numbers*", Vol. 48, 2010, No. 2, 121-128.

[3] Piero Filipponi, "*Real Fibonacci and Lucas numbers with Real subscript*", 1991.

[4] Thomas Koshy, "*Elementary and Number Theory And Applications*".

[5] Thomas Koshy, "*Fibonacci And Lucas number With Applications*".

[6] G.K. Panda and P.K. Ray, "*Co-Balancing numbers and Co-Balncers*", Internationals Journals of Mathematics and Mathematical Sciences. Vol. 2005, No. 8, 1189-1200.

[7] G.K. Panda and S.S.Rout, "*Gap Balancing Number*".

[8] G.K. Panda and S.S. Rout, "*A class of recurrent sequences exhibiting some exciting properties of Balancing numbers*", Int. J. Math. Comp. Sci., 6(2012), 4-6.

[9] G. K. Panda, "*Some fascinating properties of balancing numbers*", to appear in Fibonacci Numbers and their Applications, Vol. 10, 2006.

[10] P.K. Ray, "*Balancing and cobalancing numbers*", Ph.D. Thesis, National Institute of Technology, Rourkela, 2009.

[11] P.K. Ray, "*Balancing sequences of matrices with applications of Algebra of Balancing numbers*", Vol. 20, 2014.

[12] R. Witula, "*Fibonacci and Lucas Numbers for real indices and some applications*", ACTA PHYSICA POLONICA A, Vol. 120, 2011.